MATH 5A - TEST 4
Spring 2024
(Chapter 3.9, 4 & 5.1)

100 POINTS

Instructions on Canvas.
Show all steps, using proper notation and simplified, exact answers)

(1) Evaluate the following integrals. (2 points each)

(a) $\int_0^8 \sqrt[3]{x}\, dx$ $\boxed{12}$

$\int_0^8 x^{1/3}\,dx = \frac{3}{4}x^{4/3}\Big]_0^8 = \frac{3}{4}\cdot 8^{4/3} = \frac{3}{4}\cdot 2^4 = \frac{3}{4}\cdot 16 = 12$

(b) $\int \sec^2(x)\,dx$ $\boxed{\tan x + C}$

(2) Find the derivative of the function $g(x) = \int_2^{x^3} \sqrt{t^2+1}\, dt$ (3 points)

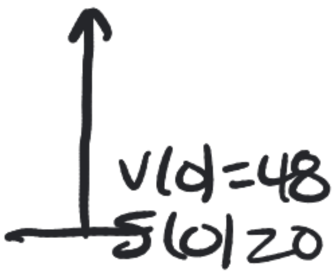
This is a function of $u = x^3$

$g'(x) = \frac{d}{du}\int_2^u (t^2+1)\,dt \quad \frac{du}{dx}$

$= \sqrt{(u^2+1)}(3x^2)$

$\boxed{= \sqrt{x^6+1}\,(3x^2)}$

(3) A ball is thrown upward from the ground with a speed of 48 ft/sec.
Find a function representing its height above the ground t seconds later. (4 points)



$a = -32$

$v = \int a\,dt = \int -32\,dt = -32t + v_0 = -32t + 48$

$s = \int v\,dt = \int (-32t + 48)\,dt = -16t^2 + 48t + s_0$

$$\boxed{s = -16t^2 + 48t}$$

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(4) Compute the following integral by making a u-substitution, **changing to and evaluating at U's limits.** (On future problems, you have your choice of how to deal with limits, but be sure to use correct notation)

$\int_0^{\pi/3} \cos(x)\,\sin^3(x)\,dx$ (7 points)

$u = \sin x$

$du = \cos x\,dx$

$x = \frac{\pi}{3} \rightarrow u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$x = 0 \Rightarrow u = \sin 0 = 0$

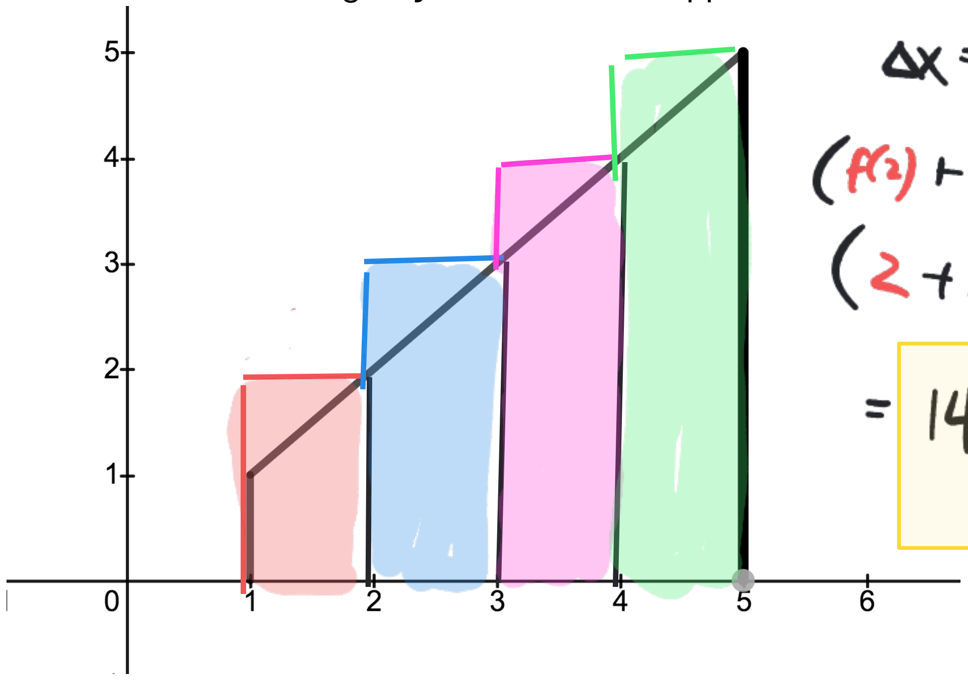
$\int_0^{\sqrt{3}/2} u^3\,du$

$\frac{1}{4}u^4\Big]_0^{\frac{\sqrt{3}}{2}} = \frac{1}{4}\cdot\frac{9}{16} = \boxed{\frac{9}{64}}$

(5) In this problem you will evaluate $\int_1^5 x\,dx$ using the 4 methods discussed in class. (20 points)
below

a) Estimate the value of $\int_1^5 x\,dx$ using n= 4 subintervals and using the right endpoints as sample points.

Draw the rectangles you used in this approximation. (3 points)



$\Delta x = \frac{5-1}{4} = 1$

$(f(2) + f(3) + f(4) + f(5))\Delta x$

$(2 + 3 + 4 + 5)$

$= \boxed{14}$

b) Integrate $\int_1^5 x\,dx$ directly, using the FTC part 2 and the antiderivative. (5 points)

$\frac{1}{2}x^2\Big]_1^5 = \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = \boxed{12}$

c) Compute $\int_1^5 x\,dx$ using the area interpretation (i.e. find the area geometrically). (2 points))

Area of trapezoid = $\left(\frac{5+1}{2}\right)(4) = \boxed{12}$

(can also break into Δ and □)

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(#5 Continued)

d) Find the exact value of $\int_1^5 x\,dx$ using the Riemann sum definition with sample points being right endpoints and the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (7 points)

$\Delta x = \frac{b-a}{n} = \frac{4}{n}$

$x_i = a + i\Delta x = 1 + i\frac{4}{n}$

$f(x_i) = 1 + i \cdot \frac{4}{n}$

$$\int_1^5 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i\cdot\frac{4}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n}\left(\sum_{i=1}^{n} 1 + \frac{4}{n}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \frac{4}{n}\left(n + \frac{4}{n}\frac{n(n+1)}{2}\right)$$

$$\boxed{= \lim_{n\to\infty}\left(4 + \frac{8n(n+1)}{n^2}\right) = 12}$$

e) Should we expect all the answers, in (a)-(d) to be the same? Why/Why not? (3 points)

Part a is an approximation, so does not necessarily match.
Parts b - d are exact, however, so should match

(6) Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(a) $\int_{-1/2}^{0} \frac{1}{\sqrt{1-2x}}\,dx$

$u = 1-2x$
$du = -2dx$

$x = 0 \Rightarrow u = 1$
$x = -1/2 \Rightarrow u = 2$

$-\frac{1}{2}\int_{2}^{1} u^{-1/2}\,du$

$-\frac{1}{2}\cdot 2u^{1/2}\Big]_{2}^{1}$

$= -u^{1/2}\Big]_{2}^{1} = \boxed{-1+\sqrt{2}}$

(b) $\int_{-1}^{1} 4x\sqrt{x^2+1}\,dx = \boxed{0}$ since $4x\sqrt{x^2+1}$ is an odd function

(6) (contd) Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(c) $\int \frac{4x^6 - x}{2x^3} dx$

$= \int \left(2x^3 - \frac{1}{2x^2}\right) dx$

$= \int \left(2x^3 - \frac{1}{2}x^{-2}\right) dx$

$= \frac{1}{2}x^4 + \frac{1}{2}x^{-1} + C$

$= \boxed{\frac{1}{2}x^4 + \frac{1}{2x} + C}$

(d) $\int x^3 \sqrt{x^2 - 3}\, dx$

$u = x^2 - 3 \qquad x^2 = u + 3$

$du = \cancel{2x}dx \Rightarrow \frac{1}{2}du = x\,dx$

$\int x^2 \cdot x \cdot \sqrt{x^2 - 3}\, dx$

$\int (u+3)\sqrt{u}\ \frac{1}{2}du$

$\frac{1}{2}\int \left(u^{3/2} + 3u^{1/2}\right) du$

$\frac{1}{2}\left(\frac{2}{5}u^{5/2} + 2u^{3/2}\right) + C$

$\boxed{\frac{1}{5}(x^2 - 3)^{5/2} + (x^2 - 3)^{3/2} + C}$

(6) cont'd Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(e) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

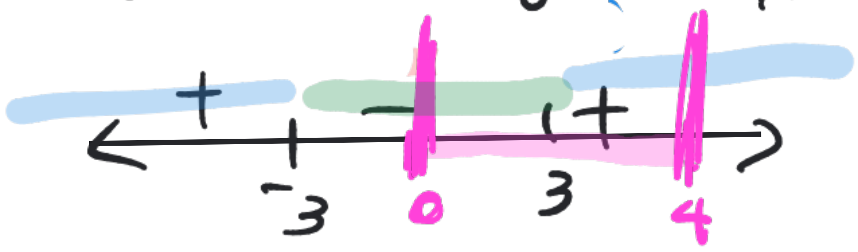
$2\int \cos u\, du$

$2\sin u + C$

$\boxed{2\sin\sqrt{x} + C}$

(f) $\displaystyle\int_0^4 \left|x^2 - 9\right| dx$

Consider the sign of $x^2 - 9$



$$|x^2 - 9| = \begin{cases} x^2 - 9 & \text{if } x \le -3 \text{ or } x \ge 3 \\ -(x^2 - 9) & \text{if } -3 < x < 3 \end{cases}$$

so

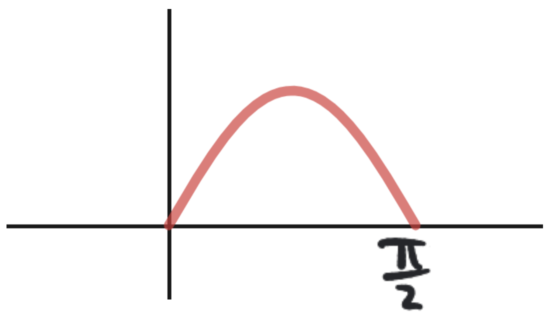
$$\int_1^4 |x^2 - 9|\,dx = \int_0^3 -(x^2 - 9)\,dx + \int_3^4 (x^2 - 9)\,dx$$

$$= -\frac{x^3}{3} + 9x \Big]_0^3 + \frac{1}{3}x^3 - 9x \Big]_3^4$$

$$= 18 + \frac{64}{3} - 36 - (-18)$$

$$= \boxed{\frac{64}{3}}$$

(7) Find the area under one "hump" of the sine curve $y = \sin(2x)$ (8 points)



$$\int_0^{\pi/2} \sin 2x \, dx$$

$u = 2x$

$du = 2dx$

$$\frac{1}{2}\int_0^{\pi} \sin u \, du = -\frac{1}{2}\cos u \Big]_0^{\pi}$$

$$= -\frac{1}{2}(\cos\pi - \cos 0)$$

$$= -\frac{1}{2}(-2)$$

$$= \boxed{1}$$

(6) Given the region bounded by the graphs of $y = x^2$, and $y = 2x+3$ (12 points)

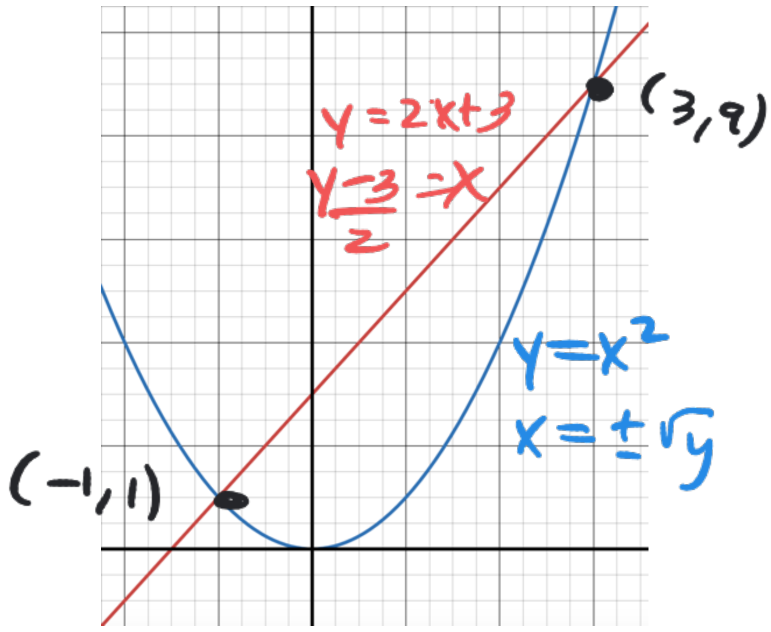
(a) Find the intersection points

$x^2 = 2x+3$

$x^2 - 2x - 3 = 0$

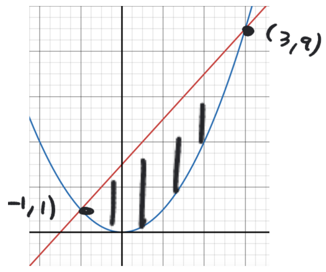
$(x-3)(x+1) = 0$

$x = 3, -1$



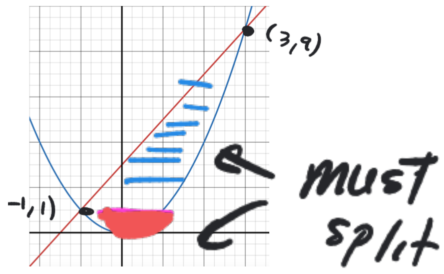
(b) Set up, but do not evaluate, an integral expression to find the area by integrating with respect to x.

$$\int_{-1}^{3} (2x+3 - x^2)\,dx$$



(c) Set up, but do not evaluate, an integral expression to find the area by integrating with respect to y.

$$\int_0^1 (-\sqrt{y} - \sqrt{y})\,dy + \int_1^9 \left(\sqrt{y} - \frac{y-3}{2}\right)dy$$



(d) Find the area by evaluating one of the integrals above.

$$\int_{-1}^{3} (2x+3 - x^2)\,dx = x^2 + 3x - \frac{1}{3}x^3\Big]_{-1}^{3}$$

$$= 9 - \left(-\frac{5}{3}\right)$$

$$= \frac{27}{3} + \frac{5}{3} = \boxed{\frac{32}{3}}$$

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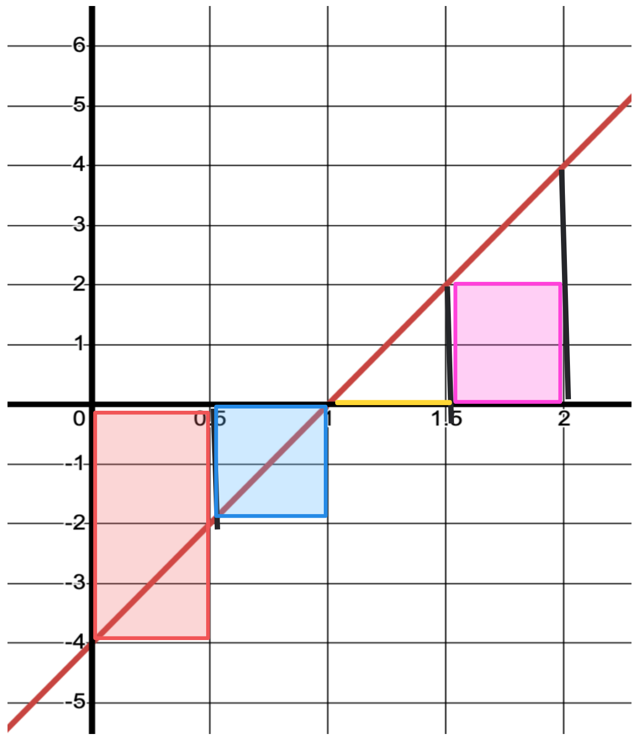
100 POINTS

Instructions on Canvas.
Show all steps, using proper notation and simplified, exact answers)

(1) In this problem you will evaluate $\int_0^2 (4x-4)\,dx$ using the 4 methods below, as discussed in class.

a) Using actual functional values (not estimates from the graph), estimate the value of $\int_0^2 (4x-4)\,dx$ using n= 4 subintervals and using the *left* endpoints as sample points. Draw the rectangles you used in this approximation. $\Delta x = \frac{1}{2}$ (3 points)

$$A \approx \left(f(0) + f\left(\tfrac{1}{2}\right) + f(1) + f\left(\tfrac{3}{2}\right)\right)\Delta x$$
$$= (-4 + -2 + 0 + 2)\,\tfrac{1}{2}$$
$$= -2$$

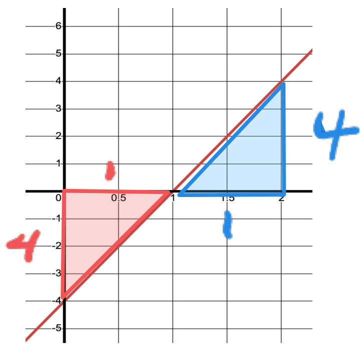


b) Integrate $\int_0^2 (4x-4)\,dx$ directly, using the FTC part 2 and the antiderivative. (5 points)

$$\int_0^2 (4x-4)\,dx = 2x^2 - 4x \Big]_0^2 = 0$$

c) Compute $\int_0^2 (4x-4)\,dx$ using the geometric area interpretation (2 points))

Area Above − Area below
$\frac{1}{2}\cdot 1\cdot 4 - \frac{1}{2}(1)(4)$
$2 - 2$
0



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d) Find the exact value of $\int_0^2 (4x-4)\,dx$ using the Riemann sum definition with sample points being right endpoints and the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (7 points)

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + i\frac{2}{n}$

$f(x_i) = 4(i \cdot \frac{2}{n}) - 4$

$$\int_0^2 (4x-4)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(4\left(i\cdot\frac{2}{n}\right)-4\right)}_{F(x_i)} \underbrace{\frac{2}{n}}_{\Delta x}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{16}{n^2} i - \frac{8}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{16}{n^2} \sum i - \frac{8}{n} \sum 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{16}{n^2} \frac{n(n+1)}{2} - \frac{8}{n} n\right)$$

$$= \lim_{n\to\infty} \left(\frac{8(n+1)}{n} - 8\right)$$

$$= \lim_{n\to\infty} \left(8 + \frac{8}{n} - 8\right)$$

$$= 0$$

e) Should we expect all the answers, in (a)-(d) to be the same? Why/Why not? (3 points)

Answers b,c,d should match because they all give the exact value of the integral. Part (a), however, is just an estimate

(2) Evaluate the following integrals. (3 points each)

(a) $\int_0^8 \sqrt[3]{x}\,dx = \int_0^8 x^{1/3}\,dx = \frac{3}{4}x^{4/3} = \frac{3}{4}\cdot 8^{4/3} = \frac{3}{4}\cdot 2^4 = \boxed{12}$

(b) $\int (6x^2 - 5x + 3)\,dx = \boxed{2x^3 - \frac{5}{2}x^2 + 3x + C}$

(c) $\int \sin(5x)\,dx$ $\quad u = 5x$, $du = 5dx$

$\frac{1}{5}\int \sin u\, du$

$-\frac{1}{5}\cos u + C = \boxed{-\frac{1}{5}\cos(5x) + C}$

(3) Find the derivative of the function g(x) = $\int_x^3 \cos^2 t\,dt = -\int_3^x \cos^2 t\,dt$ (3 points)

$g'(x) = \frac{d}{dx}\left(-\int_3^x \cos^2 t\,dt\right) = -\frac{d}{dx}\int_3^x \cos^2 t\,dt = -\cos^2 x$

(4) Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(a) On this problem only, you MUST make a u-substitution and change to U's limits. On subsequent definite integrals you can choose to switch to u's limits or not, but use proper notation.

$\int_0^{\pi/6} \cos^3(x)\,\sin(x)\,dx$

$u = \cos x$ $\quad x = \frac{\pi}{6} \rightarrow u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$du = -\sin x$ $\quad x = 0 \quad u = \cos 0 = 1$

$$\int_0^{\pi/6} \cos^3(x)\,\sin(x)\,dx = -\int_1^{\sqrt{3}/2} u^3\,du = -\frac{1}{4}u^4\Big]_1^{\sqrt{3}/2} = -\frac{1}{4}\left(\left(\frac{\sqrt{3}}{2}\right)^4 - (1)^4\right)$$

$$= -\frac{1}{4}\left(\frac{9}{16} - 1\right) = -\frac{1}{4}\left(\frac{-7}{16}\right) = \boxed{\frac{7}{64}}$$

(b) $\int x^2\left(x^3+2\right)dx$

$\int (x^5 + 2x^2)\,dx$

$$\boxed{\frac{1}{6}x^6 + \frac{2}{3}x^3 + C}$$

(4) cont'd Evaluate the following integrals. Give simplified, exact answers. (7 points each)

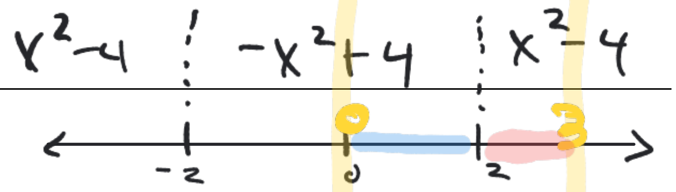
(c) $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} 4x\sin(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$$\int_0^{\pi/4} 2\cos u\,du = 2\sin u\Big]_0^{\pi/4} = 2\sin\frac{\pi}{4} = \boxed{\sqrt{2}}$$

(d) $\displaystyle\int_0^3 \left| x^2 - 4 \right| dx$



Need to rewrite $|x^2-4|$ as a piecewise defined funct.

$$|x^2-4| = \begin{cases} x^2-4 & \text{when } x^2-4 \geq 0 \Rightarrow (-\infty,-2] \cup [2,\infty) \\ -(x^2-4) & \text{when } x^2-4<0 \Rightarrow (-2,2) \end{cases}$$

$$\int_0^2 -x^2+4\,dx + \int_2^3 (x^2-4)\,dx$$

$$= -\frac{1}{3}x^3+4x\Big]_0^2 + \frac{1}{3}x^3-4x\Big]_2^3$$

$$= -\frac{8}{3}+8 + 9-12-\left(\frac{8}{3}-8\right)$$

$$= \boxed{23/3}$$

(4) cont'd Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(e) $\int \frac{x^3}{(x^4+7)^2}dx$

$u = x^4 + 7$
$du = 4x^3 dx$

$= \frac{1}{4}\int \frac{1}{u^2}du$

$= \frac{1}{4}\int u^{-2}du$

$= -\frac{1}{4}u^{-1} + C$

$= \frac{-1}{4u} + C = \frac{-1}{4(x^4+7)} + C$

(f) $\int x^2\sqrt{x+5}\,dx$

$u = x+5$ $\quad x = u - 5$
$du = dx$

$\int (u-5)^2\sqrt{u}\,du$

$\int (u^2 - 10u + 25)\sqrt{u}\,du$

$\int \left(u^{5/2} - 10u^{3/2} + 25u^{1/2}\right)du$

$= \frac{2}{7}u^{7/2} - 10\cdot\frac{2}{5}u^{5/2} + 25\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{2}{7}(x+5)^{7/2} - 4(x+5)^{5/2} + \frac{50}{3}(x+5)^{3/2} + C$

(4) cont'd Evaluate the following integrals. Give simplified, exact answers. (7 points each)

(g) $\int_{-3}^{3} 4x\sqrt{x^2+1}\,dx = 0$ $f(x) = 4x\sqrt{x^2+1}$ is an odd function

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(5) A ball is thrown upward from the ground with a speed of 112 ft/sec. $v(0)=112$
(a) Find its height above the ground t seconds later
(b). Find the maximum height. (give units) (7 points) $s(0)=0$

$a = -32 \text{ ft/sec}^2$

$v = \int a\,dt = -32t + v_0 = -32t + 112$

$s = \int v\,dt = -16t^2 + 112t + s_0$

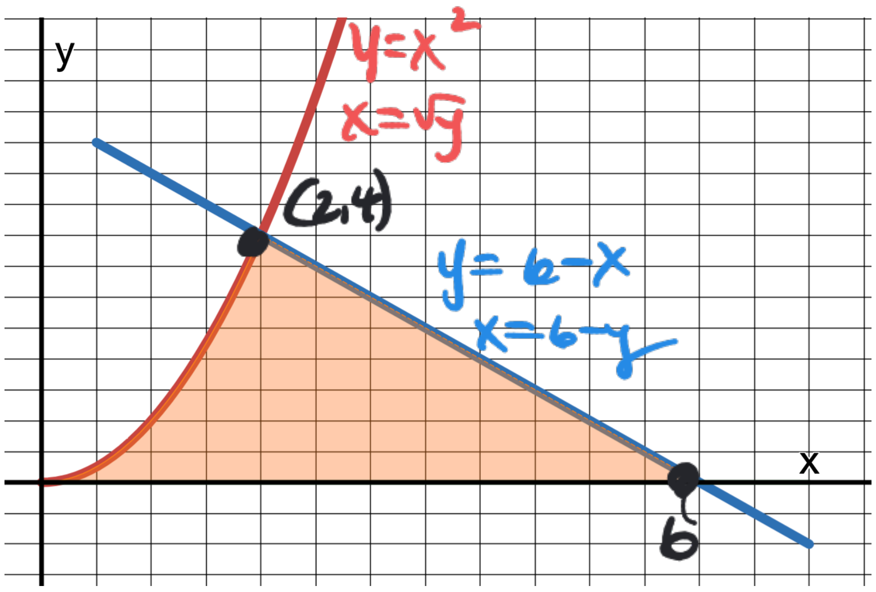
$\boxed{s(t) = -16t^2 + 112t}$

Max height when $s'(t) = v(t) = 0$ $-32t + 112 = 0$

$t = \frac{112}{32} = \frac{7}{2}$

$s\left(\frac{7}{2}\right) = \boxed{196 \text{ ft max ht.}}$

(6) Given the shaded region shown, bounded by the graphs of $y=x^2$, $y= - x+6$, and the x axis, (12 points)



Intersection

$y=x^2$
$y=6-x$ $\Rightarrow$ $x^2=6-x$ $\quad x^2+x-6=0$

$(x+3)(x-2)=0$

$x=2$

(b) Set up, but do not evaluate, an integral expression to find the area by integrating with respect to x.

$$\int_0^2 x^2\,dx + \int_2^6 (6-x)\,dx$$

(c) Set up, but do not evaluate, an integral expression to find the area by integrating with respect to y.

$$\int_0^4 \left(6-y - \sqrt{y}\right)dy$$

(d) Find the area by evaluating one of the above.

$$\int_0^4 \left(6-y - \sqrt{y}\right)dy = 6y-\tfrac{1}{2}y^2-\tfrac{2}{3}y^{3/2}\Big]_0^4$$

$$= 24 - 8 - \frac{16}{3}$$

$$= 16 - \frac{16}{3} = \frac{32}{3}$$